

MINIMUM CODEGREE THRESHOLD FOR ($K_4^3 - e$)-FACTORS

ALLAN LO AND KLAS MARKSTRÖM

ABSTRACT. Given hypergraphs H and F , an F -factor in H is a spanning subgraph consisting of vertex disjoint copies of F . Let $K_4^3 - e$ denote the 3-uniform hypergraph on 4 vertices with 3 edges. We show that for $\gamma > 0$ there exists an integer n_0 such that every 3-uniform hypergraph H of order $n > n_0$ with minimum codegree at least $(1/2 + \gamma)n$ and $4|n$ contains a $(K_4^3 - e)$ -factor. Moreover, this bound is asymptotically the best possible and we further give a conjecture on the exact value of the threshold for the existence of a $(K_4^3 - e)$ -factor. Therefore, all minimum codegree thresholds for the existence of F -factors are known asymptotically for 3-uniform hypergraphs F on 4 vertices.

1. INTRODUCTION

Given hypergraphs H and F , an F -factor (or a *perfect F -tiling* or a *perfect F -matching*) in H is a spanning subgraph consisting of vertex disjoint copies of F . Clearly, if H contains an F -factor then $|F|$ divides $|H|$. A k -uniform hypergraph, k -graph for short, is a pair $H = (V(H), E(H))$, where $V(H)$ is a finite set of vertices and $E(H) \subset \binom{V(H)}{k}$. Often we write V instead of $V(H)$ when it is clear from the context. For a k -graph H and an l -set $T \in \binom{V}{l}$, let $\deg(T)$ be the number of $(k-l)$ -sets $S \in \binom{V}{k-l}$ such that $S \cup T$ is an edge in H , and let $\delta_l(H)$ be the *minimum l -degree* of H , that is, $\delta_l(H) = \min\{\deg(T) : T \in \binom{V}{l}\}$. Define $t_l^k(n, F)$ to be the smallest integer d so that every k -graph of order n with $\delta_l(H) \geq d$ contains an F -factor. If n is not divisible by $|F|$, then $t_l^k(n, F) = \binom{n-l}{k-l}$. Hence, we always assume that $|F|$ divides n .

For graphs (that is, 2-graphs), a classical theorem of Hajnal and Szemerédi [6] states that $t_1^2(n, K_t) = (t-1)n/t$. Furthermore, $t_1^2(n, F)$ is known up to an additive constant for every 2-graph F , see [11]. For graphs F , there is a large body of research on $t_1^2(n, F)$, for surveys see [10, 15].

In the case of hypergraphs ($k \geq 3$), only a few values of $t_l^k(n, F)$ are known. Note that when F is a single edge K_k^k , a K_k^k -factor is equivalent to a perfect matching. Rödl, Ruciński and Szemerédi [14] proved that

$$t_{k-1}^k(n, K_k^k) = \frac{n}{2} - k + \epsilon, \text{ where } \epsilon \in \{3/2, 2, 5/2, 3\}.$$

Date: November 28, 2011.

Key words and phrases. Hypergraph, 3-graph, factorization, minimum codegree.

For $k > l \geq 1$, Kühn and Osthus [10] and independently Hán, Person and Schacht [7] conjectured that

$$t_l^k(n, K_k^k) = \left(\max \left\{ \frac{1}{2}, 1 - \left(1 - \frac{1}{k} \right)^{k-l} \right\} + o(1) \right) \binom{n}{k}.$$

This conjecture has been verified for various cases of k and l . We recommend [13] for a survey in $t_l^k(n, K_k^k)$.

Here, we focus to the case when $k = 3$, $l = 2$ and $|F| = 4$. Let K_4^3 be the complete 3-graph on 4 vertices. The authors [12] showed that $t_2^3(n, K_4^3) = (3/4 + o(1))n$, and independently Keevash and Mycroft [8] determine the exact value of $t_2^3(n, K_4^3)$ for n sufficiently large. For $1 \leq i \leq 3$, let $K_4^3 - ie$ be the unique 3-graph on 4 vertices with $(4-i)$ edges. Kühn and Osthus [9] showed that $t_2^3(n, K_4^3 - 2e) = (1/4 + o(1))n$, and the exact value was determined by Czygrinow, DeBiasio and Nagle [3]. Let A and B be set of $n/4 - 1$ vertices and $3n/4 + 1$ respectively. By considering the 3-graph H such that $V(H) = A \cup B$ and every edge meets A , we can deduce that $t_2^3(n, K_4^3 - 3e) > n/4 - 1$. Moreover,

$$n/4 \leq t_2^3(n, K_4^3 - 3e) \leq t_2^3(n, K_4^3 - 2e) = \begin{cases} n/4 & \text{if } n/4 \text{ is odd} \\ n/4 + 1 & \text{if } n/4 \text{ is even.} \end{cases}$$

In this paper, we investigate $t_2^3(n, K_4^3 - e)$, the only remaining case for 3-graphs on 4 vertices. It is easy to show that $t_2^3(4, K_4^3 - e) = 1$. Also, we know that $t_2^3(8, K_4^3 - e) = 4$ by a computer search. For $n \geq 12$, we give the following lower bound on $t_2^3(n, K_4^3 - e)$.

Proposition 1.1. *For integers $n \geq 8$ with $4|n$*

$$t_2^3(n, K_4^3 - e) \geq \begin{cases} n/2 & \text{if } n \not\equiv 0 \pmod{3} \\ n/2 - 1 & \text{if } n \equiv 0 \pmod{3}. \end{cases}$$

We show that the inequality above is indeed asymptotically sharp.

Theorem 1.2. *Given a constant $\gamma > 0$, there exists an integer $n_0 = n_0(\gamma)$ such that for $n \geq n_0$ and $4|n$*

$$t_2^3(n, K_4^3 - e) \leq (1/2 + \gamma)n.$$

We further conjecture that equality holds in Proposition 1.1.

Conjecture 1.3. *For integers $n \geq 8$ with $4|n$*

$$t_2^3(n, K_4^3 - e) = \begin{cases} n/2 & \text{if } n \not\equiv 0 \pmod{3} \\ n/2 - 1 & \text{if } n \equiv 0 \pmod{3}. \end{cases}$$

2. NOTATIONS AND PRELIMINARIES

For the remainder of the paper, we will only consider 3-graphs unless stated otherwise. For simplicity, we write K_4 and K_4^- for K_4^3 and $K_4^3 - e$ respectively. We refer to the set $\{1, \dots, a\}$ as $[a]$ for $a \in \mathbb{N}$.

For a 3-graph H and a vertex set $U \subset V(H)$, $H[U]$ is the subgraph of H induced by the vertices of U . We write v to mean the set $\{v\}$ when it is clear from the context. Let V_1, \dots, V_l be a partition of $V(H)$. We say that an edge $v_1 v_2 v_3$ is of type $V_{i_1} V_{i_2} V_{i_3}$ if $v_j \in V_{i_j}$ for $j \in [3]$ and denote the number of edges of type $V_{i_1} V_{i_2} V_{i_3}$ by $e(V_{i_1} V_{i_2} V_{i_3})$. Similarly, we define *types* for K_4^- . Given a 3-set T , we set $L(T)$ to be the set of vertices v such that $H[T \cup v]$ contains a K_4^- . For an edge e , we write $L(e)$ to mean $L(V(e))$.

Proposition 2.1. *Let H be a 3-graph of order n . Then, for every edge e , $|L(e)| \geq (3\delta_2(H) - n)/2$.*

Proof. Let $e = xyz$. Denote n_i to be the number of vertices v in exactly i neighbourhoods of $\{x, y\}$, $\{x, z\}$ and $\{y, z\}$. Note that $\sum n_i = n$ and $\sum i n_i \geq 3\delta_2(H)$. Thus, $2n_3 + n_2 \geq 3\delta_2(H) - n$. If a vertex v is in at least two neighbourhoods of $\{x, y\}$, $\{x, z\}$ and $\{y, z\}$, then $H[\{x, y, z, v\}]$ contains a K_4^- . Thus, the proposition follows. \square

The *Turán number* of K_4^- , $ex(n, K_4^-)$, is the maximum number of edges in a K_4^- -free 3-graph of order n . Currently, it is known that $(2/7 + o(1))\binom{n}{3} \leq ex(n, K_4^-) \leq (0.2871 + o(1))\binom{n}{3}$, where the lower bound is due to Frankl and Füredi [5] and the upper bound is due to Baber and Talbot [1]. If H is a 3-graph of order n with $e(H) > ex(n, K_4^-) + cn^3$, then we have the ‘supersaturation’ phenomenon discovered by Erdős and Simonovits [4].

Theorem 2.2 (Supersaturation). *For every constant $c > 0$, there exists a constant $c' > 0$ such that every 3-graph H of order n with $e(H) > ex(n, K_4^-) + cn^3$ contains at least $c'n^4$ copies of K_4^- .*

Corollary 2.3. *There exists a constant $c > 0$ such that every 3-graph H of order n with $e(H) > 0.3\binom{n}{3}$ contains at least cn^4 copies of K_4^- .*

Given an integer $i \geq 1$ and vertices $x, y \in V(H)$, we say that the vertex set $S \subset V(H)$ is an (x, y) -connector of length i if $S \cap \{x, y\} = \emptyset$, $|S| = 4i - 1$ and both $H[S \cup x]$ and $H[S \cup y]$ contain K_4^- -factors. Given an integer $i > 0$ and a constant $\eta > 0$, two vertices x and y are (i, η) -close to each other if there exists at least ηn^{4i-1} (x, y) -connectors of length i . We denote by $\tilde{N}_{i, \eta}(x)$ the set of vertices y that are (i, η) -close to x . The subset $U \subset V$ is said to be (i, η) -closed in H if every two vertices in U are (i, η) -close to each other. Moreover, H is said to be (i, η) -closed if $V(H)$ is (i, η) -closed in H . If η is known from context, we simply write i -closed and $\tilde{N}_i(x)$ for (i, η) -close and $\tilde{N}_{i, \eta}(x)$ respectively. For $X, Y \subset V$, a triple (x, y, S) is a (X, Y) -bridge of length i if $x \in X$, $y \in Y$ and S is an (x, y) -connector of length i . If $u \in X \cap Y$, we say (u, u, \emptyset) is an (X, Y) -bridge of length 0.

We study some basic properties of (i, η) -closeness.

Proposition 2.4. *Let $i > 0$ be an integer and let $\eta, \epsilon > 0$ be constants. Let H be a 3-graph of order n sufficiently large. Suppose that $|\tilde{N}_{i,\eta}(x)| \geq \epsilon n$ for a vertex $x \in V$. Then, $\tilde{N}_{i,\eta}(x) \subset \tilde{N}_{i+1,\eta'}(x)$ for some constant $\eta' > 0$.*

Proof. Let $y \in \tilde{N}_i(x)$ and $m = 4i - 1$. To prove the proposition, it is enough to show that y is $(i + 1, \eta')$ -close to x for some $\eta' > 0$. There are at least ηn^m (x, y) -connectors S of length i . Fix an (x, y) -connector S of length i . Let $z \in \tilde{N}_i(x) \setminus (S \cup \{x, y\})$. There are at least ηn^m (x, z) -connectors S' of length i . Moreover, the number of S' containing a vertex in $S \cup y$ is at most $(m + 1)n^{m-1} < \eta n^m/2$. Hence, there are at least $\eta n^m/2$ (x, z) -connector S' with $S' \cap (S \cup y) = \emptyset$. Since $H[S' \cup z]$ contains a K_4^- -factor, there is a 3-set $T \subset S'$ such that $z \in L(T)$. By an averaging argument, the number of K_4^- U vertex disjoint from $S \cup \{x, y\}$ is at least

$$\frac{\eta n^m/2}{n^{m-3}} \times \frac{\epsilon n - m - 2}{4} > \eta \epsilon n^4/16.$$

Recall that S is an (x, y) -connector of length i , so $S \cup U$ is an (x, y) -connector of length $i + 1$. Note also that there are

$$\frac{\eta n^m/2 \times \eta \epsilon n^4/16}{\binom{m+4}{4}} > \eta' n^{m+4}$$

such choices $S \cup U$ for some constant $\eta' > 0$. Hence, y is $(i + 1, \eta')$ -close to x . \square

Lemma 2.5. *Let $i_X, i_Y > 0$ and $i \geq 0$ be integers and let $\eta_X, \eta_Y, \eta, \epsilon > 0$ be constants. Let H be a 3-graph of order n sufficiently large with vertices $x, y \in V$. Suppose there are at least ϵn^{4i+1} copies of (X, Y) -bridges of length i , where $X = \tilde{N}_{i_X, \eta_X}(x)$ and $Y = \tilde{N}_{i_Y, \eta_Y}(y)$. Then, x and y are $(i_X + i_Y + i, \eta_0)$ -close to each other for some $\eta_0 > 0$. In particular, if $|X \cap Y| \geq \epsilon n$, then x and y are $(i_X + i_Y, \eta)$ -close to each other for some $\eta > 0$.*

Furthermore, if X and Y are (i_X, η_X) -closed and (i_Y, η_Y) -closed in H and $|X|, |Y| \geq \epsilon n$, then $X \cup Y$ is $(i_X + i_Y + i, \eta)$ -closed in H .

Proof. Let $i_0 = i_X + i_Y + i$ and let $\eta_0 > 0$ be a constant sufficiently small. Let $m_0 = 4i_0 - 1$, $m = 4i - 1$, $m_X = 4i_X - 1$ and $m_Y = 4i_Y - 1$. There are at most $(m + 2)n^{m+1} < \epsilon n^{m+2}$ copies of (X, Y) -bridges (x', y', S) of length i with $\{x, y\} \cap (S \cup \{x', y'\}) \neq \emptyset$. Hence, the number of (X, Y) -bridges (x', y', S) with $x' \in X \setminus (S \cup \{x, y\})$ and $y' \in Y \setminus (S \cup \{x, y\})$ is at least $\epsilon n^{m+2}/2$. Fix one such (X, Y) -bridge (x', y', S) . Since $x' \in X \setminus x$, the number of (x, x') -connectors S_X of length i_X such that $S_X \cap (S \cup \{x, x', y, y'\}) = \emptyset$ is at least

$$\eta_X n^{m_X} - (m + 4)n^{m_X-1} \geq \eta_X n^{m_X}/2$$

and fix one such S_X . Similarly, the number of (y, y') -connectors S_Y of length i_Y such that $S_Y \cap (S \cup S_X \cup \{x, x', y, y'\}) = \emptyset$ is at least

$$\eta_Y n^{m_Y} - (m_X + m + 4)n^{m_Y-1} \geq \eta_Y n^{m_Y}/2$$

and fix one such S_Y . Set $S_0 = S_X \cup S_Y \cup S \cup \{x', y'\}$. Note that S_0 is an (x, y) -connector of length i_0 . Moreover, there are at least

$$\frac{1}{\binom{m_0}{m, 1, 1, m_X, m_Y}} \times \frac{\epsilon n^{m+2}}{2} \times \frac{\eta_X n^{m_X}}{2} \times \frac{\eta_Y n^{m_Y}}{2} \geq \eta n^{m_0}$$

distinct S_0 , so x and y are (i_0, η_0) -close to each other. The second assertion holds as (z, z, \emptyset) is an (X, Y) -bridge of length 0 for $z \in X \cap Y$. Finally, the last assertion holds by Proposition 2.4. \square

We now state the absorption lemma for K_4^- -factors, which is a special case of Lemma 1.1 in [12].

Lemma 2.6 (Absorption lemma [12]). *Let $i > 0$ and $\eta > 0$ be an integer and a constant. Then, there is an integer n_0 satisfying the following: Suppose that H is a 3-graph of order $n \geq n_0$ and H is (i, η) -closed, then there exists a vertex subset $U \subset V(H)$ of size $|U| \leq \eta^4 n / 2^6$ such that $H[U \cup W]$ contain K_4^- -factors for every vertex set $W \subset V \setminus U$ of size $|W| \leq \eta^8 n / 2^{11}$ with $|W| + |U| \equiv 0 \pmod{4}$.*

3. A LOWER BOUND ON $t_2^3(n, K_4^3 - e)$

In this section, our aim is to prove Proposition 1.1. First we need the following simple proposition.

Proposition 3.1. *For $n \geq 5$, there exists a K_4^- -free 3-graph H of order n with $\delta_2(H) = 1$.*

Proof. We are going to prove by induction on n . For $n = 5$, we consider the 3-graph on vertex set $\{v_1, \dots, v_5\}$ with edge set $\{v_1 v_2 v_3, v_1 v_2 v_4, v_1 v_2 v_5, v_3 v_4 v_5\}$. For $n = 6$, Frankl and Füredi (Example 1 in [5]) presented a K_4^- -free 3-graph H of order 6 with $\delta(H) \geq 2$. Hence, we may assume that $n \geq 7$ and there exists a K_4^- -free 3-graphs H' of order $(n - 2)$ with $\delta_2(H') \geq 1$. Add two new vertices x and y to $V(H')$ and new edges xyz for $z \in V(H')$. Call the resultant 3-graph H . Note that $\delta_2(H) = 1$ and H is K_4^- -free. \square

Now, we are going to bound $t_2^3(n, K_4^-)$ from below, thereby proving Proposition 1.1.

Proof of Proposition 1.1. For $n = 8$, we consider the 3-graph H with vertex set $\{v_1, \dots, v_8\}$ and edge set

$$\begin{aligned} E(H) = & \left(\binom{\{v_1, \dots, v_6\}}{3} \setminus \{v_i v_5 v_6 : i = 2, 3, 4\} \right) \cup \{v_i v_7 v_8 : i = 1, 2, 3\} \\ & \cup \{v_1 v_5 v_7, v_2 v_5 v_7, v_3 v_4 v_7, v_3 v_6 v_7, v_1 v_3 v_8, v_1 v_6 v_8, v_2 v_4 v_8, v_3 v_5 v_8, v_4 v_5 v_8\} \\ & \cup \{v_1 v_2 v_i, v_1 v_4 v_i, v_4 v_6 v_i, v_5 v_6 v_i : i = 7, 8\}. \end{aligned}$$

Note that $\delta_2(H) = d(v_7v_8) = 3$ and H does not contain K_4^- -factor. Thus, we may assume that $n > 8$.

For integers $a, b > 0$, let $A = \{v_1, \dots, v_a\}$ and $B = \{w_1, \dots, w_b\}$ be two disjoint vertex sets. We define a 3-graph $H_{a,b}$ on the vertex set $A \cup B$ as follows:

- (1) $H_{a,b}[A]$ is empty.
- (2) $H_{a,b}[B]$ is a complete 3-graph,
- (3) $a_{i_1}a_{i_2}b_j$ is an edge in $H_{a,b}$ if and only if $j \notin \{i_1, i_2\}$,
- (4) $a_ib_{j_1}b_{j_2}$ is an edge in $H_{a,b}$ if and only if $i \in \{j_1, j_2\}$.

Note that $\delta_2(H_{a,b}) = \min\{b-2, b, b-1, a-1\}$ by considering $\deg(a_1a_2)$, $\deg(b_1b_2)$, $\deg(a_1b_1)$, $\deg(a_1b_2)$ respectively. Since every K_4^- in $H_{a,b}$ contains exactly 0 or 3 vertices of A , $H_{a,b}$ does not contain a K_4^- -factor if $a \not\equiv 0 \pmod{3}$. If $a \geq 5$, define $H_{a,b}^*$ to be the resulting 3-graph of $H_{a,b}$ by embedding a K_4^- -free on A such that $\delta_2(H_{a,b}^*[A]) \geq 1$. Therefore, we have $\delta_2(H_{a,b}^*) \geq \min\{a-1, b-1\}$. Also, $H_{a,b}^*$ does not contain a K_4^- -factor if $a \not\equiv 0 \pmod{3}$.

If $n \equiv 0 \pmod{3}$, then $t_2^3(n, K_4^-) > n/2 - 2$ by considering $H_{n/2-1, n/2+1}$. If $n \not\equiv 0 \pmod{3}$ and $n > 8$, then $t_2^3(n, K_4^-) > n/2 - 1$ by considering $H_{n/2, n/2}^*$. \square

Remark 3.2. *Actually, to show that $t_2^3(n, K_4^-) \geq n/2 - 1$ for $n \equiv 0 \pmod{3}$, we could consider $H_{n/2+1, n/2-1}^*$ instead of $H_{n/2-1, n/2+1}$. In fact, there is a third and probably the simplest construction, which also gives the same result. Let A and B be vertex sets of sizes $n/2 - 1$ and $n/2 + 1$ respectively. Define H' to be the 3-graphs on $A \cup B$ such that every edge contain even number of vertices in A . Note that $\delta_2(H) = n/2 - 2$ and every K_4^- contains either 0 or 3 vertices of A . Since $|A| \not\equiv 0 \pmod{3}$, H' does not contain a K_4^- -factor.*

4. AN UPPER BOUND ON $t_2^3(n, K_4^3 - e)$

In the next theorem, we study the relationship between $\delta_2(H)$ and the number of the vertex disjoint copies of K_4^- in H . Note that $|H|$ is not assumed to be divisible by 4 in the hypothesis.

Theorem 4.1. *Let $l \leq (n - 13)/4$ be an integer. Let H be a 3-graph of order n with $\delta_2(H) > (n + 2l - 2)/3$. Then, there exists at least l vertex disjoint copies of K_4^- in H .*

Proof. Let \mathcal{T} be a set of vertex disjoint copies of K_4^- and edges in H . Let \mathcal{T}_1 and \mathcal{T}_2 be the set of K_4^- and edges of \mathcal{T} respectively. If $|\mathcal{T}_1| \geq l$, then we are done. Hence, we may assume that $|\mathcal{T}_1| < l$ for all \mathcal{T} . We define the weighting $w(\mathcal{T})$ of \mathcal{T} to be $w(\mathcal{T}) = 5|\mathcal{T}_1| + 2|\mathcal{T}_2|$. We assume that \mathcal{T} is chosen such that $w(\mathcal{T})$ is maximal.

First, we are going to show that $|\mathcal{T}_2| < 4$. Suppose the contrary, so there are 4 disjoint edges $e_1, e_2, e_3, e_4 \in \mathcal{T}_2$. Note that if $v \in L(e_i)$ for some $1 \leq i \leq 4$, then $v \in V(\mathcal{T}_1)$. Otherwise, $\mathcal{T}' = (\mathcal{T} \setminus \{e_i, e_0\}) \cup \{V(e_i) \cup v\}$ contradicts

the maximality of $w(\mathcal{T})$, where e_0 is the edges in \mathcal{T} contained v if it exists. By Proposition 2.1, $|L(e_i)| \geq (3\delta_2(H) - n)/2 > l - 1$ for $i \in [4]$. Thus, there exists $S = \{v_1, v_2, v_3, v_4\} \in \mathcal{T}_1$ such that $\sum_{i \in [4]} |L(e_i) \cap S| \geq 5$. Without loss of generality, we may assume by the König-Egerváry Theorem (see [2] Theorem 8.32) that $v_1 \in L(e_1)$ and $v_2 \in L(e_2)$. Set $\mathcal{T}' = (\mathcal{T} \setminus \{S, e_1, e_2\}) \cup \{V(e_1) \cup v_1, V(e_2) \cup v_2\}$. Note that

$$w(\mathcal{T}') = w(\mathcal{T}) - (5 + 2 + 2) + (5 + 5) = w(\mathcal{T}) + 1,$$

a contradiction. Thus, we have $|\mathcal{T}_2| < 4$.

Note that

$$|V \setminus V(\mathcal{T})| \geq n - 4|\mathcal{T}_1| - 3|\mathcal{T}_2| \geq n - 4(l - 1) - 9 = n - 4l - 5 \geq 8.$$

Let $x_1, \dots, x_4, y_1, \dots, y_4$ be distinct vertices in $V \setminus V(\mathcal{T})$. Since $w(\mathcal{T})$ is maximal, $N(x_i y_i) \subset V(\mathcal{T})$. If $\sum_{i \in [4]} |N(x_i y_i) \cap V(\mathcal{T}_2)| > 4|\mathcal{T}_2|$, there exists an edge $e \in \mathcal{T}_2$ such that $\sum_{i \in [4]} |N(x_i y_i) \cap V(e)| \geq 5$. By the König-Egerváry Theorem, we may assume that $x_1 y_1 v_1$ and $x_2 y_2 v_2$ are edges for distinct vertices $v_1, v_2 \in V(e)$. Hence, $w(\mathcal{T}') = w(\mathcal{T}) + 2$, where $\mathcal{T}' = \mathcal{T} \setminus e \cup \{x_1 y_1 v_1, x_2 y_2 v_2\}$, a contradiction. Therefore, $\sum_{i \in [4]} |N(x_i y_i) \cap V(\mathcal{T}_2)| \leq 4|\mathcal{T}_2|$. Recall that $|\mathcal{T}_2| \leq 3$ and so

$$\sum_{i \in [4]} |N(x_i y_i) \cap V(\mathcal{T}_1)| \geq 4\delta_2(H) - 12 > 8|\mathcal{T}_1|$$

There exists $S = \{v_1, v_2, v_3, v_4\} \in \mathcal{T}_1$ such that $\sum |N(x_i y_i) \cap S| \geq 9$. Again by the König-Egerváry Theorem, we may assume without loss of generality that $x_i y_i v_i$ is an edge for $i \in [3]$. Set

$$\mathcal{T}' = \mathcal{T} \setminus S \cup \{x_1 y_1 v_1, x_2 y_2 v_2, x_3 y_3 v_3\}.$$

Note that $w(\mathcal{T}') - w(\mathcal{T}) \geq 3 \times 2 - 5 = 1$, a contradiction. This complete the proof of theorem. \square

Next, we are going to prove Theorem 1.2. We proceed by the absorption technique of Rödl, Ruciński and Szemerédi [14]. We require the following lemma, where is proven in Section 5.

Lemma 4.2. *Let $\gamma > 0$ and let H be a 3-graph of order n sufficiently large with $\delta_2(H) \geq (1/2 + \gamma)n$. Then, H is (i, η) -closed for some integer i and constant $\eta > 0$.*

Proof of Theorem 1.2. Let $\gamma > 0$ and let H be a 3-graph H of order n sufficiently large with $4|n$ and $\delta_2(H) \geq (1/2 + \gamma)n$. In order to prove Theorem 1.2, it is enough to show that H contains a K_4^- -factor. By Lemma 4.2, H is (i, η) -closed for some i and $\eta > 0$. We may further take η to be sufficiently small ($\eta^4/2^6 < \gamma$ would do). Let U be the vertex set given by Lemma 2.6 and so $|U| \leq \eta^4 n / 2^6$. Let $H' = H[V(H) \setminus U]$. Note that

$$\delta_{k-1}(H') \geq (1/2 + \gamma - \eta^4/2^6)n \geq n'/2$$

where $n' = n - |U|$. There exists a family \mathcal{T} of vertex disjoint copies of K_4^- in H' covering all but at most 16 vertices by Theorem 4.1. Let $W = V(H') \setminus V(\mathcal{T})$, so $|W| \leq 16$. By Lemma 2.6, there exists a K_4^- -factor \mathcal{T}' in $H[U \cup W]$. Thus, $\mathcal{T} \cup \mathcal{T}'$ is a K_4^- -factor in H . \square

5. PROOF OF LEMMA 4.2.

Let $\gamma > 0$ and let H be a 3-graph of order n sufficiently large with $\delta_2(H) \geq (1/2 + \gamma)n$. Our aim is to show that H is (i, η) -closed for some i and $\eta > 0$ proving Lemma 4.2. Its proof is divided into the following steps. First we show that we can partition $V(H)$ into at most 3 vertex classes such that each class is $(\lceil 4/\gamma \rceil + 2, \eta)$ -closed in H and of size at least $n/4$. If there is only one vertex class, then we are done. When there are two or three vertex classes, we show that H is (i', η') -closed using Lemma 5.4 and Lemma 5.6 respectively for some integer i' and constant $\eta' > 0$.

Recall that $\tilde{N}_{i,\eta}(v)$ is the set of vertices that are (i, η) -closed to v . First, we show that the size of $\tilde{N}_{1,\gamma^2/12}(v)$ is at least $(1/4 + \gamma)n$ for every $v \in V$.

Proposition 5.1. *Let $\gamma > 0$ and let H be a 3-graph of order $n > 8/\gamma$ with $\delta_2(H) \geq (1/2 + \gamma)n$. Then, for $v \in V$ there are at least $(1/4 + \gamma)n$ vertices y such that y is $(1, \gamma^2/12)$ -close to v .*

Proof. Write $\delta = \delta_2(H)$ and $V' = V \setminus v$. Let $\{x, y\} \in N(v)$, i.e. vxy is an edge. Note that there are at least $\delta(n-1)/2 \geq n^2/4$ such pairs. For $z \in N(xy) \cap N(vx)$, $H[\{v, x, y, z\}]$ contains a K_4^- . Since $|N(xy) \cap N(vx)| \geq 2\gamma n$, there are at least $\gamma n^3/6$ edges e such that $v \in L(e)$.

Let G be a bipartite graph with the following properties. The vertex classes of G are V' and E' , where E' is a set of edges e such that $v \in L(e)$. For $y \in V'$ and $e \in E'$, $\{y, e\}$ is an edge in G if and only if $y \in L(e)$. Note that $|E'| \geq \gamma n^3/6$. For $e \in E'$

$$d^G(e) = |L(e) \setminus v| \geq (1/4 + 3\gamma/2)n - 1 > (1/4 + 11\gamma/8)n$$

by Proposition 2.1. We claim that there are more than $(1/4 + \gamma)n$ vertices $y \in V'$ with $d^G(y) \geq \gamma|E'|/2$. Indeed, it is true or else we have

$$(1/4 + 11\gamma/8)n|E'| < e(G) \leq \gamma|E'|/2 \times (3/4 - \gamma)n + |E'|(1/4 + \gamma)n,$$

a contradiction. Note that $y \in V'$ is $(1, d^G(y)/n^3)$ -close to v , so the proposition follows. \square

We are going to partition V into at most three classes such that each class is of size at least $(1/4 + \gamma)n$ and $(\lceil 4/\gamma \rceil + 2, \eta)$ -closed in H for some $\eta > 0$.

Lemma 5.2. *Let $\gamma > 0$ and let H be a 3-graph of order n with $\delta_2(H) \geq (1/2 + \gamma)n$. Then, there exist a constant $\eta > 0$ and a vertex partition of V into at most three classes such that each class W is $(\lceil 4/\gamma \rceil + 2, \eta)$ -closed in H and $|W| \geq (1/4 + 3\gamma/4)n$.*

Proof. Throughout this proof, $\eta_1, \dots, \eta_{\lceil 4/\gamma \rceil + 2}$ are assumed to be a decreasing sequence of strictly positive sufficiently small constants. We write i -close to mean (i, η_i) -close. If $|\tilde{N}_2(v)| \geq (1 + \gamma)n/2$ for all $v \in V$, then $|\tilde{N}_2(v) \cap \tilde{N}_2(u)| \geq \gamma n$ for $u, v \in V$. Thus, H is 4-closed by Lemma 2.5. Hence, we may assume that there exists a vertex v such that $|\tilde{N}_2(v)| < (1 + \gamma)n/2$. Let U be the set of vertices $u \in \tilde{N}_1(v)$ such that

$$|\tilde{N}_1(u) \cap \tilde{N}_2(v)| \geq (1/4 + \gamma/3)n.$$

Claim 5.3. *The size of U is at least $(1 + 3\gamma)n/4$ and U is 2-closed in H .*

Proof of claim. Note that if $|\tilde{N}_1(w) \cap \tilde{N}_1(v)| \geq \gamma^2 n/6$ for $w \in V \setminus v$, then $w \in \tilde{N}_2(v)$ by Lemma 2.5. Thus, for each $w \notin \tilde{N}_2(v)$,

$$|\tilde{N}_1(v) \cap \tilde{N}_1(w)| < \gamma^2 n/6.$$

Therefore, by summing over all $w \notin \tilde{N}_2(v)$, we have

$$\sum_{u \in \tilde{N}_1(v)} |\tilde{N}_1(u) \setminus \tilde{N}_2(v)| = \sum_{w \notin \tilde{N}_2(v)} |\tilde{N}_1(v) \cap \tilde{N}_1(w)| < \gamma^2 n^2/6. \quad (1)$$

Since $|\tilde{N}_1(u')| \geq (1/4 + \gamma)n$ for $u' \in V$ by Proposition 5.1, for $u' \in \tilde{N}_1(v) \setminus U$

$$|\tilde{N}_1(u') \setminus \tilde{N}_2(v)| = |\tilde{N}_1(u')| - |\tilde{N}_1(u') \cap \tilde{N}_2(v)| > 2\gamma n/3.$$

Therefore, by summing over $u' \in \tilde{N}_1(v) \setminus U$ and (1), we have

$$2\gamma n |\tilde{N}_1(v) \setminus U|/3 \leq \sum_{u' \in \tilde{N}_1(v) \setminus U} |\tilde{N}_1(u') \setminus \tilde{N}_2(v)| \leq \sum_{u \in \tilde{N}_1(v)} |\tilde{N}_1(u) \setminus \tilde{N}_2(v)| < \gamma^2 n^2/6.$$

Again recall that $|\tilde{N}_1(v)| \geq (1/4 + \gamma)n$ by Proposition 5.1, so $|U| \geq (1 + 3\gamma)n/4$ as desired. Furthermore, for $u, u' \in U$, we have

$$|\tilde{N}_1(u) \cap \tilde{N}_1(u')| \geq |\tilde{N}_1(u) \cap \tilde{N}_2(v)| + |\tilde{N}_1(u') \cap \tilde{N}_2(v)| - |\tilde{N}_2(v)| \geq \gamma n/6$$

as $|\tilde{N}_2(v)| < (1 + \gamma)n/2$. Hence, u and u' are 2-close to each other by Lemma 2.5. \square

Set $U_0 = U$. For an integer $i > 0$, we define U_i to be the set of vertices $u' \notin W_{i-1}$ such that $|\tilde{N}_1(u') \cap W_{i-1}| \geq 2^{i-2}\gamma n$, where $W_{j'}$ is the set $\bigcup_{j=0}^{j'} U_j$. By Lemma 2.5 and an induction on i , we deduce that $H[W_i]$ is $(i+2)$ -closed in H . Let i_0 be the minimal integer such that $|U_{i_0}| < \gamma n/4$. Since U_0, U_1, \dots , are disjoint sets, $1 \leq i_0 \leq \lceil 4/\gamma \rceil$. If $W_{i_0} = V(H)$, then H is $(i_0 + 2)$ -closed and so H is $(\lceil 4/\gamma \rceil + 2)$ -closed by Proposition 2.4. Thus, we may assume that $V(H) \neq W_{i_0}$. Note that $|W_{i_0}| \geq |U| \geq (1 + 3\gamma)n/4$. For $w \notin W_{i_0}$, we have

$$\begin{aligned} |\tilde{N}_1(w) \setminus W_{i_0}| &\geq |\tilde{N}_1(w)| - |\tilde{N}_1(w) \cap W_{i_0-1}| - |U_{i_0}| \\ &\geq (1/4 + \gamma)n - \gamma n/4 - \gamma n/4 = (1/4 + \gamma/2)n. \end{aligned}$$

Let $V' = V \setminus W_{i_0}$. Note that $|V'| \leq 3n/4$ and $|\tilde{N}_1(u) \cap V'| \geq (1/4 + \gamma/2)n$ for $u \in V'$. Thus, we are done by repeating the whole argument at most twice by replacing V with V' . \square

To prove Lemma 4.2, it is sufficient to consider the case when there are two or three partition classes satisfying the conditions in Lemma 5.2. First, we consider the case when there are exactly two partition classes as its proof will form the framework for the case when there are three partition classes.

Lemma 5.4. *Let $i_X, i_Y > 0$ be integers and let $\eta_X, \eta_Y, \gamma > 0$ be constants. Let H be a 3-graph of order n sufficiently large with $\delta_2(H) \geq (1/2 + \gamma)n$. Suppose that V is partitioned into X and Y with $|X|, |Y| \geq n/4$. Furthermore, suppose that X and Y are (i_X, η_X) -closed and (i_Y, η_Y) -closed in H respectively. Then H is (i_0, η) -closed for some integer $i_0 \leq 3 \max\{i_X, i_Y\} + 1$ and constant $\eta > 0$.*

Proof. Write $\delta = \delta_2(H)$. Without loss of generality, we may assume that $|X| \leq |Y|$. Let $c_1, c_2, c_3, c_4, \epsilon_1, \epsilon_2, \epsilon'_2, \epsilon_3, \epsilon'_3, \epsilon_4, \epsilon_5, \epsilon'_5 > 0$ be constants sufficiently small satisfying the following six inequalities:

$$\begin{aligned} \epsilon_1 &< \min\{\epsilon_2, \epsilon_3\}, \quad c_1 + \epsilon_2 < c_2 < c_3\epsilon'_3, \quad \max\{4\epsilon'_2, 2\epsilon_1 + \epsilon'_3\} < 3\gamma \\ 2c_1 &< c_3 < \min\{c_4\epsilon_4/2 - \epsilon_3, 2^{-11}\epsilon'_5 - \epsilon_3\}, \quad \epsilon_5 \leq \gamma/384, \quad \epsilon'_5 < 1/10. \end{aligned}$$

In addition, throughout this proof, η_1, η_2, \dots are assumed to be a decreasing sequence of strictly positive sufficiently small constants. Recall that an (X, Y) -bridge of length i is a triple (x, y, S) such that $x \in X$, $y \in Y$ and S is an (x, y) -connector of length i . By Lemma 2.5, to prove the lemma it is enough to show that there are at least ϵn^{4i+1} (X, Y) -bridges of length i for some $\epsilon > 0$. Recall that $L(e)$ is the number of K_4^- contained the edge e and $|L(e)| \geq (1/4 + \gamma)n$ by Proposition 2.1. We now divide into various cases depending on the type of e and $L(e)$. For each case, we are going to show that there are many (X, Y) -bridges and so we prove Lemma 5.4.

Case 1 : There exist $c_1 n^3$ edges e such that $|L(e) \cap X| \geq \epsilon_1 n$ and $|L(e) \cap Y| \geq \epsilon_1 n$. For each such edge e , $(x, y, V(e))$ is (X, Y) -bridge for $x \in L(e) \cap X$ and $y \in L(e) \cap Y$. Therefore, there are at least $c_1 \epsilon_1^2 n^5$ (X, Y) -bridge of length 1.

Case 2 : There exist $c_2 n^4$ copies T of K_4 such that $|T \cap X| = 2 = |T \cap Y|$. There are at least $(c_2 - \epsilon_2) n^3$ edges e of type XXY contained in at least $\epsilon_2 n$ copies of these K_4 . Otherwise, the number of these K_4 is at most

$$(c_2 - \epsilon_2) n^3 \times n + (1 - c_2 + \epsilon_2) n^3 \times \epsilon_2 n < c_2 n^4,$$

a contradiction. Note that for each such edge e , $|L(e) \cap Y| \geq \epsilon_2 n$. By Case 1, we may assume that there are at least $(c_2 - \epsilon_2 - c_1) n^3$ edges e of type XXY contained in at least $\epsilon_2 n$ copies of these K_4 with $|L(e) \cap X| \leq \epsilon_1 n$. Fix one such edge $xx'y$ and let $y' \in Y$ such that $H[\{x, x', y, y'\}]$ is a K_4 . Note that there are $(c_2 - \epsilon_2 - c_1) \epsilon_2 n^4 / 2$ choices for x, x', y and y' .

Claim 5.5. *One of $L(xx'y) \cap X$, $L(xx'y') \cap X$, $L(xyy') \cap Y$, $L(x'yy') \cap Y$ is of size at least $\epsilon'_2 n$.*

Proof of claim. Suppose that the claim is false. Since $|N(xy) \cap X| \geq \delta - |N(xy) \cap Y|$ and $|N(x'y) \cap X| \geq \delta - |N(x'y) \cap Y|$,

$$\begin{aligned} |N(xx') \cap X| + |N(xy) \cap X| + |N(x'y) \cap X| - |X| &\leq 2|L(xx'y) \cap X| \leq 2\epsilon'_2 n. \\ |N(xy) \cap Y| + |N(x'y) \cap Y| &\geq 2\delta - |X| - 2\epsilon'_2 n. \end{aligned} \quad (2)$$

Similarly,

$$|N(xy') \cap Y| + |N(x'y') \cap Y| \geq 2\delta - |X| - 2\epsilon'_2 n. \quad (3)$$

In addition, we have

$$2\epsilon'_2 n + |Y| \geq |N(xy) \cap Y| + |N(xy') \cap Y| + |N(yy') \cap Y|, \quad (4)$$

$$2\epsilon'_2 n + |Y| \geq |N(x'y) \cap Y| + |N(x'y') \cap Y| + |N(yy') \cap Y| \quad (5)$$

as $|L(xyy') \cap Y|, |L(x'yy') \cap Y| \leq \epsilon'_2 n$ respectively. Recall that $|X| + |Y| = n$, $|X| \leq |Y|$ and $|N(yy') \cap Y| \geq \delta - |X|$. Together with (2), (3), (4) and (5), we have

$$6\delta \leq 4|X| + 2|Y| + 8\epsilon'_2 n \leq 3n + 8\epsilon'_2 n$$

a contradiction. \square

Recall that there are $(c_2 - \epsilon_2 - c_1)\epsilon_2 n^4/2$ choices of $\{x, x', y, y'\}$. Suppose that at least $(c_2 - \epsilon_2 - c_1)\epsilon_2 n^4/8$ copies of $K_4 = \{x, x', y, y'\}$ with $|L(xx'y) \cap X| \geq \epsilon'_2 n$. Let $u \in L(xx'y) \cap X$. Note that $(u, y', \{x, x', y'\})$ is an (X, Y) -bridge. Thus, the number of (X, Y) -bridges (of length 1) is at least $(c_2 - \epsilon_2 - c_1)\epsilon_2 \epsilon'_2 n^5/24$. Therefore, we may assume without loss of generality that there are at least $(c_2 - \epsilon_2 - c_1)\epsilon_2 n^4/8$ copies of $K_4 = \{x, x', y, y'\}$ with $|L(xyy') \cap Y| \geq \epsilon'_2 n$. Let $u \in L(xyy') \cap Y$. Note that $(x', u, \{x, y, y'\})$ is an (X, Y) -bridge. Again, the number of (X, Y) -bridges is at least $(c_2 - \epsilon_2 - c_1)\epsilon_2 \epsilon'_2 n^5/24$.

Case 3 : There exist $c_3 n^3$ edges xyy' of type XYX such that $|L(xyy') \cap X| \geq \epsilon_3 n$. By Case 1, we may assume that there are at least $c_3 n^3/2$ edges xyy' of type XYX such that $|L(xyy') \cap Y| < \epsilon_1 n$. Since xyy' is an edge and $|L(xyy') \cap Y| < \epsilon_1 n$, we have

$$|N(xy) \cap Y| + |N(xy') \cap Y| + |N(yy') \cap Y| - |Y| \leq 2|L(xyy') \cap Y| < 2\epsilon_1 n.$$

Assume that $|N(xy) \cap N(xy') \cap N(yy') \cap X| \leq \epsilon'_3 n$ and so

$$|N(xy) \cap X| + |N(xy') \cap X| + |N(yy') \cap X| - 2|X| \leq \epsilon'_3 n.$$

Since $|X| + |Y| = n$ and $|X| \leq n/2 \leq |Y|$, (by combining the two inequalities above together) we have

$$3\delta \leq d(xy) + d(x'y) + d(xx') < 2|X| + |Y| + 2\epsilon_1 n + \epsilon'_3 n \leq (3/2 + 2\epsilon_1 + \epsilon'_3)n,$$

a contradiction. Thus, we have $|N(xy) \cap N(xw) \cap N(wy) \cap X| \geq \epsilon'_3 n$. Note that for each $u \in N(xy) \cap N(xw) \cap N(wy) \cap X$, $\{u, x, y, y'\}$ is a K_4 in H .

Thus, there are at least $c_3\epsilon'_3 n^4/2 \geq c_2 n^4$ copies of K_4 with two vertices in each of X and Y . Therefore, we are done by Case 2.

Case 4 : There exist $c_4 n^3$ edges $xx'y$ of type XXY such that $|L(xx'y) \cap X| \geq \epsilon_4 n$. Hence, there are at least $c_4 \epsilon_4 n^4/2$ copies of K_4^- of type $XXYY$. Since every K_4^- of type $XXYY$ contains an edge of type XYX , there are at $c_3 n^3$ edges xyy' of type XYX such that $|L(xyy') \cap X| \geq \epsilon_3 n$. Otherwise, the number of K_4^- of type $XXYY$ is at most

$$c_3 n^3 \times n + n^3 \times \epsilon_3 n < c_4 \epsilon_4 n^4/2,$$

a contradiction. Thus, we are in Case 3.

Case 5 : None of Case 1-4 holds. Recall that $n/4 \leq |X| \leq n/2 \leq |Y|$. For every pair of vertices $x, x' \in X$, $|N(xx') \cap Y| \geq \delta - |X| \geq \gamma n$ and so $e(XXY) \geq \binom{|X|}{2}(\delta - |X|) \geq \gamma n^3/32$, where we recall that $e(V_1 V_2 V_3)$ is the number of edges of type $V_1 V_2 V_3$. Similarly, $e(XYY) \geq |X||Y|(\delta - |X|)/2 \geq \gamma n^3/32$ as $|N(xy) \cap Y| \geq \delta - |X| \geq \gamma n$ for $x \in X$ and $y \in Y$. In summary,

$$e(XXY), e(XYY) \geq \gamma n^3/32.$$

Further recall that $|L(e)| \geq (1/4 + \gamma)n$ for edges e by Proposition 2.1. We must have $\gamma n^4/384 \geq \epsilon_5 n^4$ copies of K_4^- each of type $XXXX$ and $YYYY$ as neither Case 1, Case 3 nor Case 4 holds. Next, we divide into the cases depending on the number of K_4^- of types $XXXX$ and $YYYY$.

Case 5a : There are $c' n^4$ copies of K_4^- of type $XXXX$, where c' is the constant defined in Corollary 2.3. Let $m_X = 4i_X - 1$ and $m_Y = 4i_Y - 1$. Recall that there are at least $\epsilon_5 n^4$ copies of K_4^- of type $XXXY$. Pick two vertex disjoint K_4^- , $T = \{x_1, x_2, x_3, x_4\}$ of type $XXXX$ and $T' = \{x'_1, x'_2, x'_3, y'\}$ of type $XXXY$. Since x_1 is (i_X, η_X) -close to x'_1 , there exist at least

$$\eta_X n^{m_X} - 8n^{m_X-1} \geq \eta_X n^{m_X}/2$$

copies of (x_1, x'_1) -connectors S_1 with $S_1 \cap (V(T) \cup V(T')) = \emptyset$. Fix one such S_1 . Similarly, for $i = 2, 3$ we can find an (x_i, x'_i) -connectors S_i such that $S_i \cap (V(T) \cup V(T') \cup S_1) = \emptyset$ and $S_2 \cap S_3 = \emptyset$. Furthermore, there are at least $(\eta_X n^{m_X}/2)^2$ choices for the pair (S_2, S_3) . Set

$$S = S_1 \cup S_2 \cup S_3 \cup \{x_1, x_2, x_3, x'_1, x'_2, x'_3\}.$$

Note that there is a K_4^- -factor in $H[S \cup y']$ as there is a K_4^- -factor in each of $H[T]$ and $H[x'_i \cup S_i]$ for $i = 1, 2, 3$. Also, there is a K_4^- -factor in $H[S \cup x_4]$. Thus, (x_4, y', S) is an (X, Y) -bridge of length $3i_X + 1$. Moreover, there are $\epsilon_5 c' \eta_X^3 n^{3m_X+8}/(32(3m_X+8)!)$ such (X, Y) -bridges.

Case 5b : There are $c' n^4$ copies of K_4^- of type $YYYY$. We are done by a similar argument used in Case 5a.

Case 5c : Neither Case 5a nor Case 5b holds. By Corollary 2.3, we have $e(H[X]) \leq 0.3 \binom{|X|}{3}$ and $e(H[Y]) \leq 0.3 \binom{|Y|}{3}$. Thus,

$$e(XXY) \geq (\delta - 0.3|X|) \binom{|X|}{2} \text{ and } e(XYY) \geq (\delta - 0.3|Y|) \binom{|Y|}{2}.$$

For $x, x' \in X$ and $y, y' \in Y$, define $a(x, x', y, y')$ to be the number of edges in $H[\{x, x', y, y'\}]$. Note that if $a(x, x', y, y') \geq 3$, then $H[\{x, x', y, y'\}]$ contains a K_4^- . We sum $a(x, x', y, y')$ over all $x, x' \in X$ and $y, y' \in Y$, so each edge of type XXY (and XYY) is counted $|Y| - 1$ (and $|X| - 1$) times, i.e.

$$\begin{aligned} \sum a(x, x', y, y') &= (|Y| - 1)e(XXY) + (|X| - 1)e(XYY) \\ &\geq \frac{1}{2}(|X| - 1)(|Y| - 1)(\delta(|X| + |Y|) - 0.3(|X|^2 + |Y|^2)) \\ &= \frac{1}{2}(|X| - 1)(|Y| - 1)(\delta n - 0.3(|X|^2 + |Y|^2)). \end{aligned} \quad (6)$$

If $\sum a(x, x', y, y') > (2 + 4\epsilon'_5) \binom{|X|}{2} \binom{|Y|}{2}$, then there are at least $\epsilon'_5 \binom{|X|}{2} \binom{|Y|}{2} \geq 2^{-10} \epsilon'_5 n^4$ copies of 4-sets $\{x, x', y, y'\}$ such that

$$e(H[\{x, x', y, y'\}]) = a(x, x', y, y') \geq 3$$

as $|X|, |Y| \geq n/4$. Note that $H[\{x, x', y, y'\}]$ contains a K_4^- . By an averaging argument there are at least $(2^{-11} \epsilon'_5 - \epsilon_3)n^3 \geq c_3 n^3$ edges e of type XYY with $|L(e) \cap X| \geq \epsilon_3 n$. This implies that Case 3 holds, a contradiction. Thus, we may assume that $\sum a(x, x', y, y') \leq (2 + 4\epsilon'_5) \binom{|X|}{2} \binom{|Y|}{2}$. Recall that $n/4 \leq |X| = n - |Y|$ and $\delta \geq n/2$. Therefore, (6) becomes

$$\begin{aligned} (2 + 4\epsilon'_5) \binom{|X|}{2} \binom{|Y|}{2} &\geq \frac{1}{2}(|X| - 1)(|Y| - 1)(\delta n - 0.3(|X|^2 + |Y|^2)) \\ (1 + 2\epsilon'_5)|X||Y| &\geq \delta n - 0.3(|X|^2 + |Y|^2) \\ \epsilon'_5 n^2 &\geq n^2/2 - 0.3(|X|^2 + |Y|^2) - |X||Y| \\ &= n^2/10 + 0.4(|X| - n/2)^2 \geq n^2/10, \end{aligned}$$

a contradiction. This completes the proof of Lemma 5.4. \square

We now consider the case when V is partitioned into 3 classes, X', Y' and Z' such that $|X'|, |Y'|, |Z'| \geq (1/4 + \gamma)n$ and X', Y' and Z' are $(\lceil 4/\gamma \rceil + 2, \eta)$ -closed in H . Its proof is based on the proof of Lemma 5.4.

Lemma 5.6. *Let $\gamma > 0$ and let H be a 3-graph of order n with $\delta_2(H) \geq (1/2 + \gamma)n$. Suppose that $V(H)$ is partitioned into X', Y' and Z' with $|X'|, |Y'|, |Z'| \geq n/4$ and X', Y' and Z' are $(i_{X'}, \eta_{X'})$ -closed, $(i_{Y'}, \eta_{Y'})$ -closed and $(i_{Z'}, \eta_{Z'})$ -closed in H respectively. Then H is (i, η) -closed for some integer $i > 0$ and constant $\eta > 0$.*

Proof. Write $\delta = \delta_2(H)$. Let $m_{X'} = 4i_{X'} - 1$, $m_{Y'} = 4i_{Y'} - 1$ and $m_{Z'} = 4i_{Z'} - 1$. Let $c_1, c_2, c_3, c_4, \epsilon_1, \epsilon_2, \epsilon'_2, \epsilon_3, \epsilon'_3, \epsilon_4, > 0$ be constant as defined in the proof of Lemma 5.4 with an extra constant $\epsilon_0 > 0$ satisfying $c_3 + \epsilon_3 \leq \epsilon_0 \leq \gamma$.

Again, η_1, η_2, \dots are assumed to be a decreasing sequence of strictly positive sufficiently small constants.

A triple (u, v, S) is an i -bridge if it is either (X', Y') -bridge, (X', Z') -bridge or (Y', Z') -bridge of length i . If the number of i -bridges is at least ϵn^{4i+1} for some constants $\epsilon > 0$, then we may assume that without loss of generality that there are at least $\epsilon n^{4i+1}/3$ (X', Y') -bridges. Hence, $X' \cup Y'$ is $(i_{X'} + i_{Y'} + i)$ -closed in H by Lemma 2.5 and so H is i_0 -closed by Lemma 5.4 for some i_0 . Therefore, to prove the lemma it is enough to show that there exist an integer i_0 and a constant $\epsilon > 0$ such that the number of i_0 -bridges is at least ϵn^{4i_0+1} .

First, suppose that there are at least $\epsilon_0 n^4$ copies of K_4^- of each of type $X'X'Y'Z'$ and $X'Y'Y'Z'$. Hence, we can pick two vertex disjoint copies of K_4^- , $T = \{x_1, x_2, y, z\}$ of type $X'X'Y'Z'$ and $T' = \{x', y'_1, y'_2, z'\}$ of type $X'Y'Y'Z'$. Since x_1 is $(i_{X'}, \eta_{X'})$ -close to x' , there exist at least $\eta_{X'} n^{m_{X'}}/2$ copies of (x_1, x') -bridge $S_{X'}$ with $S_{X'} \cap (V(T) \cup V(T')) = \emptyset$. Fix one such $S_{X'}$. Similarly, we can find a (y, y'_1) -bridge $S_{Y'}$ and a (z, z') -bridge $S_{Z'}$ such that $S_{Y'} \cap S_{Y'} = \emptyset$ and $(S_{Y'} \cup S_{Z'}) \cap (S_{X'} \cup V(T) \cup V(T')) = \emptyset$. Furthermore, there are at least $\eta_{Y'} n^{m_{Y'}}/2$ and $\eta_{Z'} n^{m_{Z'}}/2$ choices for $S_{Y'}$ and $S_{Z'}$ respectively. Set $S = S_{X'} \cup S_{Y'} \cup S_{Z'} \cup \{x_1, x', y, y'_1, z, z'\}$. Note that (x_4, y', S) is an (X, Y) -bridge of length $i_0 = i_X + i_Y + i_Z + 1$. Moreover, there are $\epsilon_0^2 \eta_{X'} \eta_{Y'} \eta_{Z'} n^{m_0}/(32(m_0)!)$ such (X, Y) -bridges, where $m_0 = 4i_0 + 1$. Hence, we may assume without loss of generality that there are less than $\epsilon_0 n^4$ copies of K_4^- of each of type $X'Y'Y'Z'$ and $X'Y'Z'Z'$.

We now mimic the proof of Lemma 5.4 by setting $X = X'$ and $Y = Y' \cup Z'$. Note that $|X| + |Y| = n$ and $|Y| = |Y'| + |Z'| \geq n/2 \geq |X| \geq n/4$. As in proof of Lemma 5.4 we divide into different cases. Also, observe that an (X, Y) -bridge of length i is an i -bridge. Hence, the lemma is proved if we can show that there are many (X, Y) -bridges of length i in each case.

Case 1' : There exist $c_1 n^3$ edges e such that two of $|L(e) \cap X'|$, $|L(e) \cap Y'|$ and $|L(e) \cap Z'|$ is at least $\epsilon_1 n$. Let e be an edge such that $|L(e) \cap X'|, |L(e) \cap Y'| \geq \epsilon_1 n$. Then, $(x, y, V(e))$ is (X', Y') -bridge for $x \in L(e) \cap X'$ and $y \in L(e) \cap Y'$. Therefore, there are at least $\epsilon_1^2 c_1 n^5$ copies of 1-bridges.

Case 2' : There exist $c_2 n^4$ copies T of K_4 such that $|T \cap X| = 2 = |T \cap Y|$. By following the argument used in proving Case 2 in the proof of Lemma 5.4 (where we replace Case 1 with Case 1'), we deduce that there are ϵn^5 (X, Y) -bridges of length 1.

Case 3' : There exist $c_3 n^3$ edges xy_1y_2 of type $XY Y$ such that $|L(xy_1y_2) \cap X| \geq \epsilon_3 n$. By Case 1', we may assume that there are at least $c_3 n^3/2$ edges xy_1y_2 of type $XY Y$ such that $|L(xy_1y_2) \cap Y| < \epsilon_1 n$. By the same argument used in Case 3 in the proof of Lemma 5.4, we deduced that Case 2' holds and so we are done.

Case 4' : There exist $c_4 n^3$ edges x_1x_2y of type $XX Y$ such that $|L(x_1x_2y) \cap Y| \geq \epsilon_4 n$. Hence, there are at least $\epsilon_4 c_4 n^4$ copies of K_4^- with two vertices in each of X and Y . Since each such K_4^- must contain an edge of type $XY Y$, we are in Case 3' by an averaging argument.

Case 5' : None of Case 1'-4' holds. Since Case 3' does not hold, there are less than $(c_3 + \epsilon_3)n^4$ copies of K_4^- of type $XXYY$. Therefore, there are less than $(c_3 + \epsilon_3)n^4 < \epsilon_0 n^4$ copies of K_4^- of type $X'X'Y'Z'$. Recall that there are less than $\epsilon_0 n^4$ copies of K_4^- of each of type $X'Y'Y'Z'$ and $X'Y'Z'Z'$. Thus, there are less than $3\epsilon_0 n^4$ copies of K_4^- contained an edge of type $X'Y'Z'$. Since $|L(e)| \geq (1/4 + \gamma)n$ for every edge e by Proposition 2.1,

$$e(X'Y'Z') \leq 24\epsilon_0 n^3.$$

Without loss of generality, we may further assume that $|X'| \leq |Y'| \leq |Z'|$. Let $|X'| + |Y'| = \alpha n$, so $1/2 \leq \alpha \leq 2/3$. Since $(|X'| + |Y'|) + (|X'| + |Z'|) \geq 2\alpha n$ and $|X'| + |Y'| + |Z'| = n$, we have

$$|X'| \geq (2\alpha - 1)n. \quad (7)$$

Note that

$$\begin{aligned} e(X'Y'Y') &= \frac{1}{2} \left(\sum_{x \in X', y \in Y'} (d(xy) - |X'| + 1) - e(X'Y'Z') \right) \\ &\geq |X'| |Y'| (\delta - |X'| + 1)/2 - 12\epsilon_0 n^3 \\ &\geq |X'| |Y'| (\delta - |X'| + 1 - \epsilon' n)/2, \end{aligned}$$

where $\epsilon' = 96\epsilon_0$. Similarly, we have

$$e(X'X'Y') \geq |X'| |Y'| (\delta - |Y'| + 1 - \epsilon' n)/2.$$

For $x, x' \in X'$ and $y, y' \in Y'$, define $a(x, x', y, y')$ to be the number of edges in $H[\{x, x', y, y'\}]$ as before. Since there are less than $(c_3 + \epsilon_3)n^4$ copies of K_4^- of type $XXYY$, $\sum a(x, x', y, y') \leq (2 + 4c') \binom{|X'|}{2} \binom{|Y'|}{2}$, where $c' = c_3 + \epsilon_3$. Therefore,

$$\begin{aligned} \sum a(x, x', y, y') &= (|Y'| - 1)e(X'X'Y') + (|X'| - 1)e(X'Y'Y') \\ (1 + 2c') \frac{(|X'| - 1)(|Y'| - 1)}{2} &\geq \delta - (|X'| + |Y'| - 2) - \epsilon' n)(|X'| + |Y'| - 2) \\ &\quad + 2(|X'| - 1)(|Y'| - 1) \\ (1 - c')(|X'| - 1)(|Y'| - 1) &\leq (|X'| + |Y'| - 2 - \delta + \epsilon' n)(|X'| + |Y'| - 2). \end{aligned} \quad (8)$$

Recall that $|X'| + |Y'| = \alpha n$ and $|X'| \geq (2\alpha - 1)n$ by (7). Since $1/2 \leq \alpha \leq 2/3$ and $c' = c_3 + \epsilon_3 \leq \gamma$, $(\alpha - 1/2 - \gamma/2)/(1 - 2c') < (\alpha - 1/2)$. By taking $|Y'| = \alpha n - |X'|$, $|X'| = (2\alpha - 1)n$ and $\delta \geq (1/2 + \gamma)n$, (8) becomes

$$\begin{aligned} (1 - 2c')(2\alpha - 1)(1 - \alpha) &\leq (\alpha - 1/2 - \gamma/2)\alpha \\ (2\alpha - 1)(1 - \alpha) &< (\alpha - 1/2)\alpha \\ (2\alpha - 1)(2 - 3\alpha)/2 &< 0 \end{aligned}$$

a contradiction. This completes the proof of Lemma 5.6. \square

Therefore, Lemma 4.2 follows immediately from Lemma 5.2, Lemma 5.4 and Lemma 5.6.

6. CLOSING REMARKS

Clearly, we would like to know the exact value of $t_2^3(n, K_4^-)$. If Conjecture 1.3 is true, then by Remark 3.2 we know that the extremal graph is not unique. However, each of the constructions contains an induced K_4^- -free subgraph of size roughly $n/2$.

Proposition 3.1 shows that there exist K_4^- -free 3-graphs H of order $n \geq 5$ with $\delta_2(H) = 1$. In fact, there are K_4^- -free 3-graphs H of order n with $\delta_2(H) = (1/4 + o(1))n$. Take a random tournament on n vertices, let H be 3-graph on the same vertex set such that every edge in H is a directed triangle. Note that H is K_4^- -free and $\delta_2(H) = (1/4 + o(1))n$.

Question 6.1. *For $\epsilon > 0$, does all 3-graphs of order n sufficiently large with $\delta_2(H) \geq (1/4 + \epsilon)n$ contain a K_4^- ?*

Note that a 3-graph H with $\delta_2(H) \geq \gamma|H|$ contains at least $\gamma \binom{|H|}{3}$ edges. Thus, a result of Baber and Talbot [1] implies that the question above is true for $\delta_2(H) \geq (0.2871 + o(1))n$.

REFERENCES

- [1] R. Baber and J. Talbot, *Hypergraphs do jump*, Combin. Probab. Comput. **20** (2011), no. 2, 161–171.
- [2] J. A. Bondy and U. S. R. Murty, *Graph theory*, Graduate Texts in Mathematics, vol. 244, Springer, New York, 2008.
- [3] A. Czygrinow, L. DeBiasio, and B. Nagle, *Tiling 3-uniform hypergraphs with K_4^-* , 3-2e, Arxiv preprint arXiv:1108.4140 (2011).
- [4] P. Erdős and M. Simonovits, *Supersaturated graphs and hypergraphs*, Combinatorica **3** (1983), no. 2, 181–192.
- [5] P. Frankl and Z. Füredi, *An exact result for 3-graphs*, Discrete Math. **50** (1984), no. 2-3, 323–328.
- [6] A. Hajnal and E. Szemerédi, *Proof of a conjecture of P. Erdős*, Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969), North-Holland, Amsterdam, 1970, pp. 601–623.
- [7] H. Hàn, Y. Person, and M. Schacht, *On perfect matchings in uniform hypergraphs with large minimum vertex degree*, SIAM J. Discrete Math. **23** (2009), no. 2, 732–748.
- [8] P. Keevash and R. Mycroft, *A geometric theory for hypergraph matching*, Arxiv preprint arXiv:1108.1757 (2011).
- [9] D. Kühn and D. Osthus, *Loose Hamilton cycles in 3-uniform hypergraphs of high minimum degree*, J. Combin. Theory Ser. B **96** (2006), no. 6, 767–821.
- [10] ———, *Embedding large subgraphs into dense graphs*, Surveys in combinatorics 2009, London Math. Soc. Lecture Note Ser., vol. 365, Cambridge Univ. Press, Cambridge, 2009, pp. 137–167.
- [11] ———, *The minimum degree threshold for perfect graph packings*, Combinatorica **29** (2009), no. 1, 65–107.
- [12] A. Lo and K. Markström, *F-factors in hypergraphs via absorption*, Arxiv preprint arXiv:1105.3411 (2011).

- [13] V. Rödl and A. Ruciński, *Dirac-type questions for hypergraphs—a survey (or more problems for Endre to solve)*, An Irregular Mind (Szemerédi is 70), vol. 21, Bolyai Soc. Math. Studies, 2010.
- [14] V. Rödl, A. Ruciński, and E. Szemerédi, *Perfect matchings in large uniform hypergraphs with large minimum collective degree*, J. Combin. Theory Ser. A **116** (2009), no. 3, 613–636.
- [15] R. Yuster, *Combinatorial and computational aspects of graph packing and graph decomposition*, Computer Science Review **1** (2007), no. 1, 12–26.

SCHOOL OF MATHEMATICS, UNIVERSITY OF BIRMINGHAM, BIRMINGHAM, B15 2TT, UK

E-mail address: `s.a.lo@bham.ac.uk`

DEPARTMENT OF MATHEMATICS AND MATHEMATICAL STATISTICS, UMEÅ UNIVERSITY, S-901 87 UMEÅ, SWEDEN

E-mail address: `klas.markstrom@math.umu.se`